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DATA INTERPOLATION BY CUBIC SPLINES

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The cubic spline is a cubic polynomial that can be used with data sets of either fixed or variable mesh length. It is useful for interpolating data, smoothing data, estimating volume, and achieving greater precision in numerical analysis. The technique has a variety of applications in forestry and related disciplines. A FORTRAN IV computer program, available upon request, will yield a bi-cubic spline interpolation for either a variable mesh length or an equal interval grid.

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Cubic splines are a mathematical formulation of a mechanical technique used by draftsmen for many years, in which they fit a flexible strip of material to known points in order to form a smooth curve. For mathematical purposes, the known points are data values on a fixed or variable mesh and the flexible strip is approximated by a piecewise continuous function. This function is defined as a cubic polynomial in each interval between the known points, with continuous first and second derivations.

The cubic spline is useful as a data interpolation technique, as a finite element method in numerical analysis, as a data smoother and as a volume estimator. Meteorologists have used cubic splines for data smoothing[1], for telescoping a grid[2], and for evaluating derivatives in numerical analysis computations.[3] Cubic splines are of potential use in forestry and fire applications that deal with numeric data bases. Examples of such use include smoothing or interpolating digital topography, working with brush cover heights and classes, estimating volume contained between tree tops and ground level from sparse data, interpolating growth rates between known points, and in other applications dealing with a spatial or time series data base.

SINGLE CUBIC SPLINE

The basic mathematical theory for both the single cubic spline (one-dimensional) and bi-cubic spline (two-dimensional) is explained by Ahlberg and others.[4] A concise reformulation of the basic theory tailored to application follows.

If $S_d(f;x)$ is the spline of interpolation of the function $f(x)$ on the mesh d, then

$$\int_a^b \left| f''(x) \right|^2 dx =$$

$$\int_a^b \left| s_d''(f;x) \right|^2 dx + \int_a^b \left| f''(x) - s_d''(f;x) \right|^2 dx \qquad (1)$$

where primes represent differentiation. The integral on the left hand side of (1) is minimized by the spline fit and end conditions $s_d''(f;a) = s_d''(f;b) = 0$. In words, the cubic spline represents the curve with minimum curvature that will fit the given data, be continuous through the second derivative and assure the "smoothest" fit to the data.

The derivations which follow may be found in expanded form in the publication by Ahlberg and others[4]. Consider the mesh d of length b-a subdivided such that

$d: a = x_o < x_i < \ldots . < x_n = b$

where the spacing between the x_i need not be uniform. Data values at the x_i points are given by

$Z: z_o, z_i, \ldots . z_n$

and may represent digital elevation values or some other form of data on a grid. We seek a spline function $S_d(x)$ which is continuous through its second derivative ($S_d''(x)$), coincides with a cubic in each subinterval $x_{i-1} \le x \le x_i$ ($i = 1, 2, \ldots . n$) and satisfies $S_d(x_i) = z_i$ at the given points x_i. $S_d(x)$ is periodic or nonperiodic depending on whether the condition $S_d(a+) = S_d(b-)$ is satisfied or not.

We designate M_i the moment S_d'' and from the prescribed linearity of the second derivative we write

$$S''_d(x) = M_{i-1}\left(\frac{x_i - x}{h_i}\right) + M_i\left(\frac{x - x_{i-1}}{h_i}\right) \qquad (2)$$

where $h_i = x_i - x_{i-1}$ is the spacing between the known data points. Equation 2 simply restates the minimum curvature property and uses the prescribed linear change of the second derivative between known points x_i.

Integrating (2) twice and evaluating the constants yields

$$S'_d(x) = -M_{i-1}\left(\frac{(x_i - x)^2}{2h_i}\right) + M_i\left(\frac{(x - x_{i-1})^2}{2h_i}\right) +$$

$$\left(\frac{z_i - z_{i-1}}{h_i}\right) - \left(\frac{M_i - M_{i-1}}{6}\right) h_i \qquad (3)$$

$$S_d(x) = M_{i-1}\left(\frac{(x_i - x)^3}{6h_i}\right) + M_i\left(\frac{(x - x_{i-1})^3}{6h_i}\right) +$$

$$\left(z_{i-1} - \left(\frac{M_{i-1} h_i^2}{6}\right)\right)\left(\frac{x_i - x}{h_i}\right) +$$

$$\left(z_i - \left(\frac{M_i h_i^2}{6}\right)\right)\left(\frac{x - x_{i-1}}{h_i}\right)$$

Equations 3 give the slope of the spline and equations 4 the spline itself. We will continue the development here in terms of the moments, M_i, but one could just as easily develop the technique in terms of the slopes. One sided limits of (3) yield

$$S'_d(x_i-) = \left(\frac{h_i}{6} M_{i-1}\right) + \left(\frac{h_i}{3} M_i\right) + \left(\frac{z_i - z_{i-1}}{h_i}\right)$$

$$S'_d(x_i+) = -\left(\frac{h_{i+1}}{3} M_i\right) - \left(\frac{h_{i+1}}{6} M_{i+1}\right) + \left(\frac{z_{i+1} - z_i}{h_{i+1}}\right)$$

These and the slope continuity condition yield

$$\left(\frac{h_i}{6} M_{i-1}\right) + \left(\frac{h_i + h_{i+1}}{3} M_i\right) + \left(\frac{h_{i+1}}{6} M_{i+1}\right) =$$

$$\left(\frac{z_{i+1} - z_i}{h_{i+1}}\right) - \left(\frac{z_i - z_{i-1}}{h_i}\right)$$

which, for the periodic spline, give N-1 simultaneous equations for the moments $M_1, M_2, \ldots . M_n$. For the nonperiodic spline, additional, or end, conditions must be supplied. These conditions are a matter of experiment. We have found that specifying the slope at the end points (a,b) by linearly extrapolating the z values from (z_1, z_2) and (z_{n-1}, z_n) works well. Generalized end conditions are

$$2M_o + \lambda_o M_1 = C_o$$

$$\mu_n M_{n-1} + 2M_n = C_n$$

in which

$$\lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}}$$

$$\mu_i = 1 - \lambda_i \quad (i = 1, 2, \ldots . , n-1)$$

Equations 6 now become

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = 6$$

$$\frac{\left[(z_{i+1} - z_1)/h_{i+1}\right] - \left[(z_i - z_{i-1})/h_i\right]}{h_i + h_{i+1}}$$

Writing equations 7 and 8 in matrix form as

$$\begin{bmatrix} 2 & \lambda_0 & 0 & \cdots & 0 & 0 & 0 \\ \mu_1 & 2 & \lambda_1 & \cdots & 0 & 0 & 0 \\ 0 & \mu_2 & 2 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 2 & \lambda_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & \mu_{n-1} & 2 & \lambda_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \cdot \\ \cdot \\ \cdot \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_{n-2} \\ C_{n-1} \\ C_n \end{bmatrix} \qquad (10a)$$

or $AM = C$ (10b)

shows them to be a tri-diagonal system of equations which can be solved by numerical techniques. Repeated calculations on the same mesh (i.e., only the z_i change) can yield considerable time savings due to the non-variance of h_i, λ_i, and μ_i.

Use of equal interval meshes can save even more computational time. In many meteorological applications, large iterative computations are carried out on equal interval, multi-level grids. As a result much time can be saved. For an equal interval mesh, the inverse of the coefficient matrix A of equations 10 takes on simplified form. Equation 10b may be rewritten as $M = A^{-1}C$, and the moments then may be calculated by inverting the coefficient matrix A. The nxn determinant of A in terms of constant λ is

$$D_n(\lambda) = \begin{vmatrix} 2 & \lambda & 0 \cdots & & & \\ 1-\lambda & 2 & \lambda & & & \\ 0 & \lambda & 2 & & & \\ & & \cdots & 2 & 0 & 0 \\ & & & 1-\lambda & 2 & \lambda \\ & & & 0 & 1-\lambda & 2 \end{vmatrix} \qquad (11)$$

since $\lambda = \dfrac{h_i}{h_i + h_{i+1}} = 1/2$ for equal intervals, the determinant $D_n = D_n(1/2)$ yields

$$\left.\begin{aligned} D_n &= \frac{\left(\frac{1+3^{1/2}}{2}\right)^{n+1} - \left(\frac{1-3^{1/2}}{2}\right)^{n+1}}{3^{1/2}} \\ D_0 &= 1 \\ D_{-1} &= 0 \end{aligned}\right\} \qquad (12)$$

The coefficient determinant $|A|$ satisfies

$$|A| = 4D_{n-1} - 1/2D_{n-2} \qquad (13)$$

And the elements of the inverse of the coefficient matrix are

$$A^{-1}_{i,j} = \frac{(-1)^{i+j}\left(2D_{j-1} - 1/4D_{i-2}\right)\left(2D_{n-j-1} - 1/4D_{n-j-2}\right)}{2^{i-j}\,|A|}$$

$$(0 < j \leq i \leq N) \qquad (14a)$$

$$A^{-1}_{i,j} = \frac{(-1)^{i+j}\left(2D_{j-1} - 1/4D_{j-2}\right)\left(2D_{n-i-1} - 1/4D_{n-i-2}\right)}{2^{i-j}\,|A|}$$

$$(0 \leq j \leq i < N) \qquad (14b)$$

$$A^{-1}_{o,j} = \frac{(-1)^{j}\,1/2\left(2D_{n-j-1} - 1/4D_{n-j-2}\right)}{2^{j-1}\,|A|}$$

$$(0 < j \leq N) \qquad (14c)$$

$$A^{-1}_{N,j} = \frac{(-1)^{N+j}\,1/2\left(2D_{j-1} - 1/4D_{j-2}\right)}{2^{N-j-1}\,|A|}$$

$$(0 < j < N) \qquad (14d)$$

Since $M_i = \sum_{j=o}^{N} A^{-1}_{i,j}\, C_j$

we get $M_i = A^{-1}_{i,o}\left[C_o + \frac{6}{h^2}\left(Z_o + \lambda_o Z_1\right)\right] - \frac{18}{h^2}\sum_{j=o}^{N} A^{-1}_{i,j}\, Z_j$

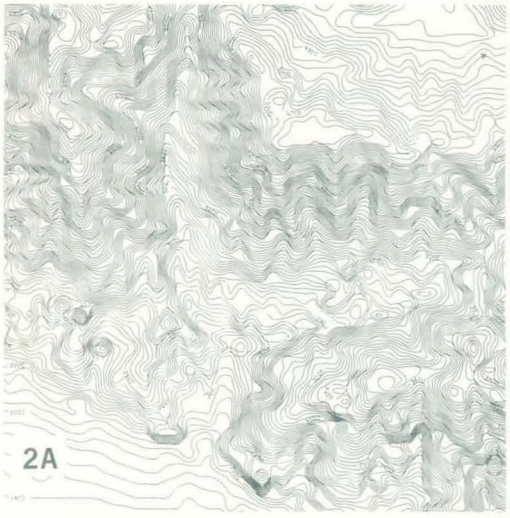
$$+ \frac{6}{h^2} Z_i + A^{-1}_{i,N}\left[C_N + \frac{6}{h^2}\left(Z_N - \mu_N Z_{N-1}\right)\right] \qquad (16)$$

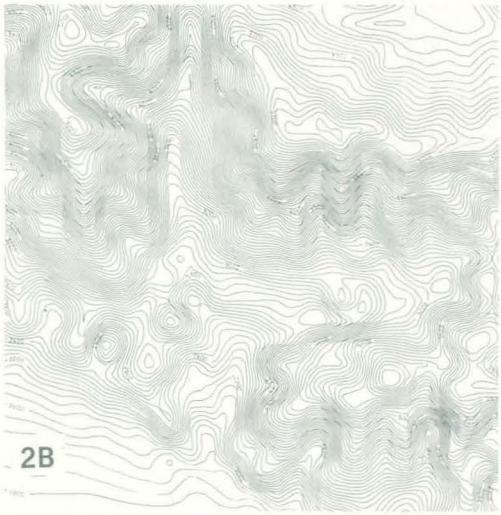
Additional shortcuts can be used when the spline is used operationally on an equal interval mesh when the Z field does not vary drastically from run to run. Since the $A^{-1}_{i,j}$ decay rapidly with distance from the principal diagonal (i.e., abs(i-j) becomes large), experimentation should yield suitable truncation methods.

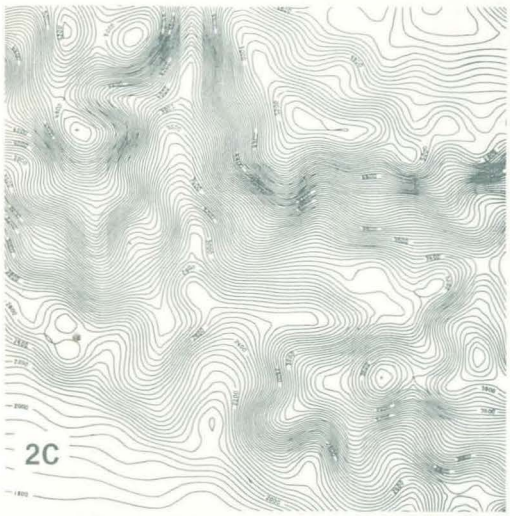
Bi-cubic Spline

We have dealt so far with the single cubic spline. For a bi-cubic spline, extension to a grid of k rows by l columns is easily accomplished by considering the calculated $S_d(x)$ values to be new z values and rotating the single cubic spline technique through 90^0. For example, consider that you wish to interpolate to a point κ, γ between rows k and k+1 and columns l and l+1. Calculate $S_d(\gamma)$ for all rows and calculate $S_d(\kappa, \gamma)$ for the interpolated values $S_d(\gamma)$ in the "γ column".

Figure 1—Section of USGS San Bernardino north quadrangle 7.5-minute topographic map shows the Devil Canyon area of the San Bernardino National Forest, in southern California.







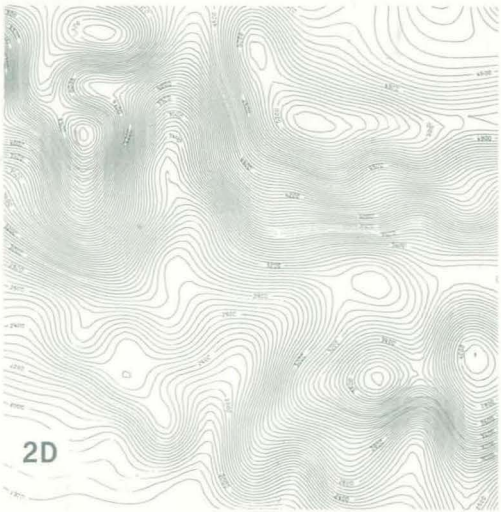


Figure 2—Computer generated plots of digitized topography for the area shown in Figure 1. The plots are based on elevation values at gridpoints on a 400-foot (122-m) mesh. The gridpoint data represents actual values (2A), values interpolated from an 800-foot mesh (2B), values interpolated from a 1200-foot mesh (2C), and values interpolated from a 1600-foot mesh (2D).

APPLICATION

The cubic spline may be used to interpolate data or to smooth data, to estimate volume or to replace or supplement finite difference approximations in numerical calculations. Before applying the spline technique for interpolation, the user must understand the nature of the available data and remember that the spline is a technique for achieving a polynomial fit with minimum curvature. If the actual data contain significant variations, or frequencies (wavelengths) that are higher than sample data collection, then the root mean square error associated with spline interpolation is not likely to be significantly better than the root mean square error associated with a linear interpolation scheme. The spline interpolation scheme will provide a minimum curvature fit, to the data, that is continuous through the second derivative. Spline interpolation is advisable, therefore, when the interpolated data are to be used in calculations which may be adversely affected by discontinuous slopes.

As an example of data interpolation and smoothing, we applied the bi-cubic spline technique to digital terrain heights on a grid. The topography of the Devil Canyon area of the San Bernardino National Forest in southern California (*fig. 1*) had been hand-digitized at a horizontal spacing of 400 feet (122 m) and an elevation resolution of 20 feet (6 m). A meteorological research network is located there and is used in the development of numerical weather models which also require topographic data. We created new data sets by using every other point (800-foot spacing) of the original grid, every third point (1200-foot spacing), and every fourth point (1600-foot spacing). The new data sets were then interpolated by bi-cubic splines to the original grid points. The root mean square errors in elevation at the interpolated points that resulted were 67 feet at the 800-foot mesh length, 93 feet at 1200 and 119 feet at 1600. *Figures 2a, b, c, and d* show computer-generated plots of the original 400-foot digitization and the three interpolations. Note how the many small ravines of 500-foot to 1000-foot width that are seen in the quadrangle map (*fig. 1*) are represented and smoothed in the four plotted digital maps (*fig. 2*). If it is assumed that a grid of mesh length Δ can resolve features with a 2Δ characteristic scale, then one can expect features of 800, 1600, 2400, 3200-foot widths to be resolved in *figs. 2a, b, c, and d.* The resolutions we achieved were actually somewhat better. Major topographic features such as the east and west forks of Devil Canyon, the Pine Flat and Camp Paivika plateaus, and Monument Peak are discernible through the 1600-foot mesh (*fig. 2d*).

Users should determine the balance between resolution and smoothing required for their particular needs and choose an appropriate interpolation interval. If variable resolution is required, the spline can telescope to higher resolution in areas of interest while maintaining lower resolution elsewhere. This technique can be applied in digitizing the data by choosing a tight digitizing interval where gradients are large and allowing looser intervals elsewhere, and then having the variable mesh length spline generate an equal mesh length grid.

A FORTRAN IV computer program, written by Julio L. Guardado, University of California, Riverside, will yield a bi-cubic spline interpolation for either a variable mesh length or an equal interval grid. Copy of the program is available on request to: Director, Pacific Southwest Forest and Range Experiment Station, P.O. Box 245, Berkeley, California 94701, Attention: Computer Services Librarian.

NOTES

[1] Fritsch, J. M. 1971. *Objective analysis of a two-dimensional data field by the cubic spline technique.* Month. Weather Rev. 99:379-386.

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